

# Revising hull and box consistency

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## Abstract

Most interval-based solvers in the constraint logic programming framework are based on either hull consistency or box consistency (or a variation of these ones) to narrow domains of variables involved in continuous constraint systems. This paper first presents HC4, an algorithm to enforce hull consistency without decomposing complex constraints into primitives. Next, an extended definition for box consistency is given and the resulting consistency is shown to subsume hull consistency. Finally, BC4, a new algorithm to efficiently enforce box consistency is described, that replaces BC3—the “original” solely Newton-based algorithm to achieve box consistency—by an algorithm based on HC4 and BC3 taking care of the number of occurrences of each variable in a constraint. BC4 is then shown to significantly outperform both HC3 (the original algorithm enforcing hull consistency by decomposing constraints) and BC3.

## 1 Introduction

Finite representation of numbers precludes computers from exactly solving continuous problems. *Interval constraint solvers* such as Prolog IV, Numerica [14], and ILOG Solver [12], tackle this problem by relying on *interval arithmetic* [10] to compute verified approximations of the solutions to constraint systems. Domains are associated to every variable occurring in the problem, and solving a particular constraint then lies in eliminating some of the values of these domains for which the constraint does not hold (inconsistency), using *local consistency techniques* and *filtering* [9].

In practice, enforced consistencies only approximate perfect local consistency since some solutions may be unrepresentable with floating-point numbers. Two worth mentioning approximate consistencies are *hull consistency* [1] and *box consistency* [2]. Most interval constraint solvers are based on either one of them. Enforcing hull consistency usually requires decomposing the user’s constraints into so-called *primitive constraints* [4].

A well known drawback of this method is that the introduction of new variables induced by the decomposition hinders efficient domain tightening. On the other hand, the original algorithm enforcing box consistency processes constraints without decomposing them but is not at best with constraints involving many variables with few occurrences. In [6], some of the authors have presented **DeclIC**, a CLP language allowing the programmer to choose the “best fitted” consistency to use for each constraint of a system; however, deferring the choice of the consistency to the user spoils the declarativity of the language.

In this paper, **HC4**, an algorithm to enforce hull consistency is first presented, that traverses repeatedly from top to bottom and conversely the tree-structured representation of constraints; consequently, decomposition of complex constraints into “primitives” is no longer needed. Next, a slightly extended definition of box consistency is given, that no longer solely relies on the *natural interval extension* of constraints and captures both the original definition of box consistency [2] and the one by Collavizza *et al.* [5]. A new algorithm (**BC4**) permitting to efficiently enforce box consistency is then given. **BC4** adapts the computation method to the number of occurrences of each variable in a constraint: domain narrowing for variables occurring only once in a constraint (which is one of the weaknesses of the “original” Newton-based method to enforce box consistency) is obtained by using **HC4**; variables occurring more than once are handled by searching “extreme quasi-zeros” using an interval Newton method as described in the original paper [2].

The rest of this paper is organized as follows: Section 2 presents the basics related to interval constraint solving. Section 3 presents hull consistency along with the usual scheme used to enforce it [4]; next, Algorithm **HC4** to enforce hull consistency without decomposing constraints is described and its properties are pointed out. Section 4 first presents the extended definition for box consistency; Algorithm **BC4** to efficiently enforce box consistency is then given. Section 5 comments results obtained on a prototype implementing **BC4**. Last, Section 6 summarizes the paper’s contribution and points out some directions for future research.

## 2 Preliminary notions

In order to model the change from continuous domains to discrete domains induced by the shift from reals to machine numbers, Benhamou and Older [3] have introduced the notion of *approximate domain* over the set of reals<sup>1</sup>  $\mathbb{R}$ : an approximate domain  $\mathcal{A}$  over  $\mathbb{R}$  is a subset of the power set of  $\mathbb{R}$ ,  $\mathcal{P}(\mathbb{R})$ , closed under intersection, containing  $\mathbb{R}$ , and for which the inclusion is a well-founded ordering. The *approximation w.r.t.  $\mathcal{A}$  of a real relation  $\rho$* ,

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<sup>1</sup>The original definition is more general but the one given here is sufficient for our purpose.

written  $\text{apx}_{\mathcal{A}}(\rho)$ , is then defined as the intersection of all the elements of  $\mathcal{A}$  containing  $\rho$ . This section focuses on two widely used approximation sets over  $\mathbb{R}$ , namely *intervals* and *unions of intervals*. The shift from reals to *intervals* is first described; interval constraints are then introduced; finally, the basics related to interval constraint solving are presented.

Let  $\mathbb{R}$  be the set of reals compactified with the infinities  $\{-\infty, +\infty\}$  in the obvious way, and  $\mathbb{F} \subset \mathbb{R}$  a finite subset of reals corresponding to *binary floating-point numbers* in a given format [8]. Let  $\mathbb{F}^\infty$  be  $\mathbb{F} \cup \{-\infty, +\infty\}$ . For every  $\mathbf{g} \in \mathbb{F}^\infty$ , let  $\mathbf{g}^+$  be the smallest element in  $\mathbb{F}^\infty$  greater than  $\mathbf{g}$ , and  $\mathbf{g}^-$  the greatest element in  $\mathbb{F}^\infty$  smaller than  $\mathbf{g}$  (with the conventions:  $(+\infty)^+ = +\infty$ ,  $(-\infty)^- = -\infty$ ,  $(+\infty)^- = \max(\mathbb{F})$ ,  $(-\infty)^+ = \min(\mathbb{F})$ ).

A *closed/open floating-point interval* is a connected set of reals whose lowest upper bound and greatest lower bound are floating-point numbers. The following notations are used as shorthands:  $[\mathbf{g}.. \mathbf{h}] \equiv \{r \in \mathbb{R} \mid \mathbf{g} \leq r \leq \mathbf{h}\}$ ,  $[\mathbf{g}.. \mathbf{h}) \equiv \{r \in \mathbb{R} \mid \mathbf{g} \leq r < \mathbf{h}\}$ , etc. Let  $\mathbb{I}_\circ$  be the set of closed/open floating-point intervals and  $\mathbb{I}_\square$  the set of closed floating-point intervals. Let  $\mathbb{U}_\circ$  be the set of unions of disjoint closed/open intervals, and  $\mathbb{U}_\square$  the restriction of  $\mathbb{U}_\circ$  to unions of disjoint closed intervals. Let  $\mathbb{I}$  (resp.  $\mathbb{U}$ ) denote the set of intervals (resp. unions of intervals) when the distinction between closed and closed/open bounds is useless. For the sake of clarity and otherwise explicitly stated, we will only consider hereafter (unions of) closed floating-point intervals.

A Cartesian product of  $n$  intervals  $\mathbf{B} = I_1 \times \dots \times I_n$  is called a *box*; a *domain*  $D$  is either an interval  $I$  or a union  $U$  of disjoint intervals. A non-empty interval  $I = [\mathbf{g}.. \mathbf{h}]$  is said to be *canonical* whenever  $\mathbf{h} \leq \mathbf{g}^+$ . An  $n$ -ary box  $\mathbf{B}$  is canonical whenever intervals  $I_1, \dots, I_n$  are canonical.

In the sequel, a real (resp. interval) constraint is an atomic formula built from a real (resp. interval)-based structure and a set of real (resp. interval)-valued variables. Given a real constraint  $c$ , let  $\rho_c$  denote the underlying relation,  $\text{Var}(c)$  be the set of variables occurring in  $c$ , and  $\text{Multiplicity}(v, c)$  the multiplicity (i.e. the number of occurrences) of the variable  $v$  in the constraint  $c$ .

An *interval extension* of  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a mapping  $F: \mathbb{I}^n \rightarrow \mathbb{I}$  such that for all  $I_1, \dots, I_n \in \mathbb{I}$ :  $r_1 \in I_1, \dots, r_n \in I_n \Rightarrow f(r_1, \dots, r_n) \in F(I_1, \dots, I_n)$ . An interval extension of a relation  $\rho \subseteq \mathbb{R}^n$  is a relation  $R \subseteq \mathbb{I}^n$  such that for all  $I_1, \dots, I_n \in \mathbb{I}$ :  $\exists r_1 \in I_1 \dots \exists r_n \in I_n$  s.t.  $(r_1, \dots, r_n) \in \rho \Rightarrow (I_1, \dots, I_n) \in R$ .

A real relation  $\rho$  may be conservatively approximated by the smallest (w.r.t. set inclusion) union of disjoint boxes  $\text{Union}(\rho) = \text{apx}_{\mathbb{U}}(\rho)$  (resp. the smallest box  $\text{Hull}(\rho) = \text{apx}_{\mathbb{I}}(\rho)$ ) containing it. In the sequel, the following approximations are used:  $\text{Union}_\circ(\rho) = \text{apx}_{\mathbb{U}_\circ}(\rho)$ ,  $\text{Union}_\square(\rho) = \text{apx}_{\mathbb{U}_\square}(\rho)$ ,  $\text{Hull}_\circ(\rho) = \text{apx}_{\mathbb{I}_\circ}(\rho)$ , and  $\text{Hull}_\square(\rho) = \text{apx}_{\mathbb{I}_\square}(\rho)$ .

An  $n$ -ary interval operation  $\blacklozenge$  is called the *natural interval extension* of an  $n$ -ary real operation  $\diamond$  if for all  $I_1, \dots, I_n \in \mathbb{I}$ :  $\blacklozenge(I_1, \dots, I_n) = \text{Hull}(\{\diamond(x_1, \dots, x_n) \mid x_1 \in I_1, \dots, x_n \in I_n\})$ . The *natural interval extension* of a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is the interval function obtained from  $f$  by replacing each con-

stant  $r$  by  $\text{Hull}(\{r\})$ , each variable by an interval variable, and each operation by its natural interval extension. Given an  $n$ -ary real constraint  $c$ , let  $\pi_k(\rho_c) = \{r_k \in \mathbb{R} \mid \exists r_1, \dots, \exists r_n \in \mathbb{R} \text{ s.t. } (r_1, \dots, r_n) \in \rho_c\}$  be the  $k$ -th projection of  $\rho_c$ .

Given a real constraint  $c(x_1, \dots, x_n)$ , a constraint solver aims at reducing the domains associated to variables  $x_1, \dots, x_n$ . This reduction process is abstracted by the notion of *constraint narrowing operators* [1] (shortened thereafter to CNOs) which are complete, contracting, and monotone functions taking as input a box and returning a box from which have been discarded some of the elements which do not belong to  $\rho_c$ .

Due to space requirements, we only give proof sketches for the properties stated hereinafter.

### 3 Hull consistency and related algorithms

Discarding all values of a box  $\mathbf{B}$  for which a real constraint  $c$  does not hold is not achievable in general. Section 3.1 presents a coarser consistency called *hull consistency* [3] consisting in computing the smallest box that contains  $\rho_c \cap \mathbf{B}$ . Section 3.2 describes HC4, a new algorithm tackling one of the drawbacks of the original algorithm used to enforce hull consistency, that is, the decomposition of the user's constraints.

#### 3.1 Hull consistency: the original scheme

The original definition of hull consistency is based on the approximate domain  $\mathbb{I}_\square$ , though it might easily be defined on  $\mathbb{I}_\circ$ :

**Definition 3.1** (hull consistency [3]). A real constraint  $c$  is said *hull consistent w.r.t. a box  $\mathbf{B}$*  if and only if  $\mathbf{B} = \text{Hull}_\square(\rho_c \cap \mathbf{B})$ .

Due to round-off errors introduced by the use of floating-point numbers, computing the interval enclosure of a real set  $\mathcal{S}$  is a difficult task in itself. Moreover, the precision of many arithmetic functions such as  $\exp$ ,  $\cos$ ... is not guaranteed by the IEEE 754 standard [8]; consequently, their actual precision is implementation dependent.

Algorithm HC3<sup>2</sup> [2, 4] partly overcomes this problem by enforcing hull consistency over a decomposition  $c_{\text{dec}}$  of simple—*primitive*—constraints rather than considering the user constraint  $c$ . For example, the constraint  $c: x + y * z = t$  might be decomposed into  $c_{\text{dec}} = \{y * z = \alpha, x + \alpha = t\}$  with the addition of the new variable  $\alpha$ .

Formally, given a real  $n$ -ary constraint  $c$  and a box  $\mathbf{B}$ , let  $\rho_c^{(k)}(\mathbf{B})$  be

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<sup>2</sup>HC3 is our own denomination for Algorithm Nar given in [2] and is justified by its very close relation to AC3.

the  $k$ -th canonical extension of  $\rho_c$  w.r.t.  $\mathbf{B}$  defined as follows [13]:

$$\rho_c^{(k)}(\mathbf{B}) = \{r_k \in \mathbb{R} \mid \exists r_1 \in I_1, \dots, \exists r_{k-1} \in I_{k-1}, \\ \exists r_{k+1} \in I_{k+1}, \dots, \exists r_n \in I_n \text{ s.t. } (r_1, \dots, r_n) \in \rho_c\}$$

A constraint  $c$  is called a *primitive constraint* on the approximate domain  $\mathcal{A}$  if and only if one can exhibit  $n$  *projection narrowing operators*  $N_c^1, \dots, N_c^n$ , defined from  $\mathcal{A}^n$  to  $\mathcal{A}$ , such that:  $\forall \mathbf{D} = D_1 \times \dots \times D_n, \forall k \in \{1, \dots, n\}: N_c^k(\mathbf{D}) = \text{apx}_{\mathcal{A}}(\rho_c^{(k)}(\mathbf{D}) \cap D_k)$ .

The CNOs for primitive constraints are implemented using *relational interval arithmetic* [4] (see an example below). The main advantage of such an approach is that computation of hull consistency can be implemented very efficiently for the set of simple constraints supported by the constraint programming system. The drawbacks are that the introduction of new variables due to the decomposition process drastically hinders domain tightening for the variables the user is interested in. As pointed out in [2], this is particularly true when the same variables appear more than once in the constraints since each occurrence of a variable  $v$  is considered as a new variable  $v'$  with the same domain as  $v$  (*dependency problem* [10]).

**Example 3.1** (A CNO for  $c: x + y = z$ ). Enforcing hull consistency for the constraint  $c: x + y = z$  and domains  $I_x, I_y$ , and  $I_z$ , is done by computing the common fixed-point included in  $\mathbf{B} = I_x \times I_y \times I_z$  of the following projection operators:  $N_c^1(\mathbf{B}) = I_x \cap (I_z \ominus I_y)$ ,  $N_c^2(\mathbf{B}) = I_y \cap (I_z \ominus I_x)$ , and  $N_c^3(\mathbf{B}) = I_z \cap (I_x \oplus I_y)$ , where  $\ominus$  and  $\oplus$  are interval extensions of  $-$  and  $+$  defined as:  $[a .. b] \oplus [c .. d] = [a + c .. b + d]$ , and  $[a .. b] \ominus [c .. d] = [a - d .. b - c]$ .

As pointed out by Van Emden [13], the projection operators  $N_c^1, N_c^2, N_c^3$ , for a constraint of the form  $x \diamond y = z$  may all exist even if the function  $\diamond$  has no inverse. For example, consider the case where  $\diamond$  stands for the multiplication over  $\mathbb{I}_{\square}$ :  $N_c^1$  computes the smallest union of intervals containing the set  $\{r_x \in I_x \mid \exists r_y \in I_y, \exists r_z \in I_z: r_x \times r_y = r_z\}$ . Hence, it is defined even when  $0 \in I_y$ .

### 3.2 HC4: a new algorithm for hull consistency

Algorithm 1 presents the HC4 algorithm which takes as input a set of constraints and a box, and narrows the variables' domains as much as possible. Algorithm HC4 is very similar to Algorithm HC3 [1] except that input constraints  $c_1, \dots, c_m$ , are user's constraints instead of decomposed constraints, and that constraint narrowing operators associated to the constraints are implemented by Algorithm HC4revise, whose description follows.

Algorithm HC4 shares the following properties with HC3: if the computation of HC4revise terminates for any constraint and any box, HC4 terminates; the algorithm is complete: the output Cartesian product of domains is a superset of the declarative semantics of the constraint system included

in the input box; the algorithm is confluent: the output is independent of the reinvoation order of constraints.

Algorithm 1: HC4 algorithm

```

HC4(in  $\{c_1, \dots, c_m\}$ ; inout  $B = I_1 \times \dots \times I_n$ )
begin
   $S \leftarrow \{c_1, \dots, c_m\}$ 
  while ( $S \neq \emptyset$  and  $B \neq \emptyset$ ) do
     $c \leftarrow$  choose one  $c_i$  in  $S$ 
     $B' \leftarrow$  HC4Revise( $c, B, \mathbb{I}_{\square}$ )
    if ( $B' \neq B$ ) then
       $S \leftarrow S \cup \{c_j \mid \exists x_k \in \text{Var}(c_j) \wedge I'_k \neq I_k\}$ 
       $B \leftarrow B'$ 
    else % HC4revise not idempotent: next call on c may further narrow domains
       $S \leftarrow S \setminus \{c\}$ 
    endif
  endwhile
end

```

These properties are proved in the same way as for HC3 (see [11]) once HC4revise has been proved to be a constraint narrowing operator (Prop. 1).

HC4 improves the compilation time (generation of primitives is useless), the solving time (no propagation needed between different primitive constraints for a given user's constraint), and the memory size.

Algorithm 2: HC4revise algorithm

```

HC4revise(in  $c = r(t_1, \dots, t_p)$ : real constraint; inout  $B$ : box;
in  $\mathcal{A}$ : approximate domain)
begin
   $D_{\mathcal{A}} \leftarrow B$ 
  foreach  $i \in \{1, \dots, p\}$  do
    ForwardEvaluation( $t_i, D_{\mathcal{A}}$ )
  endforeach
  BackwardPropagation( $c, D_{\mathcal{A}}$ )
   $B \leftarrow$  Hull $_{\square}(D_{\mathcal{A}})$ 
end

```

Algorithm 2 presents HC4revise, the new algorithm implementing the constraint narrowing operators used in HC4. Once more, note that HC4revise considers the user's constraints rather than primitives generated by decomposition: A real constraint  $r(t_1, \dots, t_p)$  is represented by an *attribute tree* where the root node contains the  $p$ -ary relation symbol  $r$ , and terms  $t_i$  are composed of nodes containing either a variable, a constant, or an operation symbol. Moreover, each node but the root contains two interval attributes

$t.fwd$  (synthesized) and  $t.bwd$  (inherited).

Given a real constraint and a box  $\mathbf{B}$ , Algorithm HC4revise proceeds in two consecutive stages:

Algorithm 3: Forward evaluation algorithm

```

ForwardEvaluation(inout  $t$ : attribute tree;
                 in  $D_{\mathcal{A}} = D_1 \times \dots \times D_n$ : Cartesian product of domains)
begin
  case ( $t$ ) of
     $\diamond(t_1, \dots, t_j)$ : % a term
      foreach  $i \in \{1, \dots, j\}$  do
        ForwardEvaluation( $t_i, D_{\mathcal{A}}$ )
      endforeach
       $t.fwd \leftarrow \diamond(t_1.fwd, \dots, t_j.fwd)$ 
     $a$ : % a constant
       $t.fwd \leftarrow \text{apx}_{\mathcal{A}}(\{a\})$ 
     $x_k$ : % a variable
       $t.fwd \leftarrow D_k$ 
  endcase
end

```

1. The *forward evaluation phase* (see Alg. 3 and Fig. 1) is a traversal of the terms from leaves to roots in order to evaluate in  $t.fwd$  the natural interval extension of every sub-term  $t$  of the constraint;
2. The *backward propagation phase* (see Alg. 4 and Fig.2) is a traversal of the tree-structured representation of the constraint from root to leaves in order to evaluate in every  $t.bwd$  a projection narrowing operator associated to the father of node  $t$ . More precisely, two kinds of node are considered:
  - given the root node  $r(t_1, \dots, t_p)$ , attributes  $t_k.bwd$  ( $k \in \{1, \dots, p\}$ ) are computed by the  $k$ -th projection narrowing operator of the constraint  $r(x_1, \dots, x_p)$ , where  $x_1 \in t_1.fwd, \dots, x_p \in t_p.fwd$ ;
  - given a term  $t_{h+1} \equiv \diamond(t_1, \dots, t_h)$ , attributes  $t_k.bwd$  ( $k \in \{1, \dots, h\}$ ) are computed by the  $k$ -th projection narrowing operator of the constraint  $x_{h+1} = \diamond(x_1, \dots, x_h)$ , where  $x_i \in t_i.fwd$  for all  $i \in \{1, \dots, h\}$ , and  $x_{h+1} \in t_{h+1}.bwd$ ; intuitively, variables  $x_i$  simulate the new ones introduced by the decomposition process.

During the backward propagation phase, the algorithm may be prematurely ended by the computation of an empty interval in some  $t.bwd$ . The constraint is then inconsistent w.r.t. the initial domains. If the computation is successful, the domain of each variable  $x_i$  in the constraint is intersected with the one in  $x_i.bwd$ .

**Note:** Backward propagation in a term is quite similar to automatic differentiation in reverse mode [7] for computing the partial derivatives of a real function: after the forward evaluation, the aim is either to evaluate the projection narrowing operators, or the partial derivatives for each node containing an operation symbol. At the end of the traversal in the tree, the domains are either intersected, or the derivatives summed.

Algorithm 4: Backward propagation algorithm

```

BackwardPropagation(inout  $t$ : attribute tree;
                   inout  $D_{\mathcal{A}} = D_1 \times \dots \times D_n$ : Cartesian product of domains)
begin
  case ( $t$ ) of
     $r(t_1, \dots, t_m)$ : % The root node
       $c \leftarrow (r(x_1, \dots, x_m))$ 
       $D'_{\mathcal{A}} \leftarrow (t_1.fwd, \dots, t_m.fwd)$ 
      foreach  $i \in \{1, \dots, m\}$  do
         $t_i.bwd \leftarrow \pi_i(\text{apx}_{\mathcal{A}}(\rho_c \cap D'_{\mathcal{A}}))$ 
        BackwardPropagation( $t_i, D_{\mathcal{A}}$ )
      endforeach
     $\diamond(t_1, \dots, t_h)$ : % A term below the root
       $c \leftarrow (\diamond(x_1, \dots, x_h) = x_{h+1})$ 
       $D'_{\mathcal{A}} \leftarrow (t_1.fwd, \dots, t_h.fwd, t.bwd)$ 
      foreach  $i \in \{1, \dots, h\}$  do
         $t_i.bwd \leftarrow \pi_i(\text{apx}_{\mathcal{A}}(\rho_c \cap D'_{\mathcal{A}}))$ 
        BackwardPropagation( $t_i, D_{\mathcal{A}}$ )
      endforeach
     $x_j$ : % A variable
       $D_j \leftarrow D_j \cap t.bwd$  % Domains intersected to take into
      % account multiple occurrences of  $x_j$ 
  endcase
end

```

**Example 3.2.** Figures 1 and 2 illustrate the computation of HC4revise over the constraint  $2x = z - y^2$ , with domains  $I_x = [0 .. 20]$ ,  $I_y = [-10 .. 10]$ , and  $I_z = [0 .. 16]$ .

Backward propagation at the root node computes  $[0 .. 40] \cap [-100 .. 16]$  (the intersection corresponds to the interpretation of the equality between intervals) in  $(\times(2, x)).bwd$  and  $(-(z, \wedge(y, 2))).bwd$ . Backward propagation at the root node of term  $\times(2, x)$  computes in  $x.bwd$  the interval  $I$  included in  $[0 .. 20]$  (result of forward evaluation) which verifies  $2 \times I_x = [0 .. 16]$ . Finally, the new domains (located in the grey nodes) are  $I_x = [0 .. 8]$ ,  $I_y = [-4 .. 4]$ , and  $I_z = [0 .. 16]$ .

**Proposition 1.** *Given an  $n$ -ary real constraint  $c$ , Algorithm HC4revise implements a constraint narrowing operator for  $c$ .*

*Proof.* Follows from the fact that HC4revise consists essentially in applying



some primitive CNOs at each node of a finite tree, and that a composition of CNOs is also a CNO.  $\square$

Let  $\text{HC4revise}^*$  be Algorithm  $\text{HC4revise}$  where, given a node  $t$ , the attribute  $t.bwd$  is not eventually intersected with  $t.fwd$ .

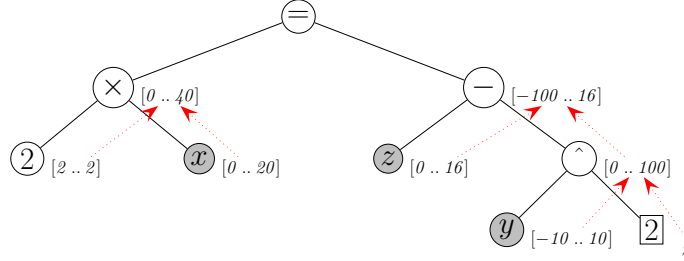


Figure 1: Annotated tree for the forward evaluation in the constraint  $2x = z - y^2$

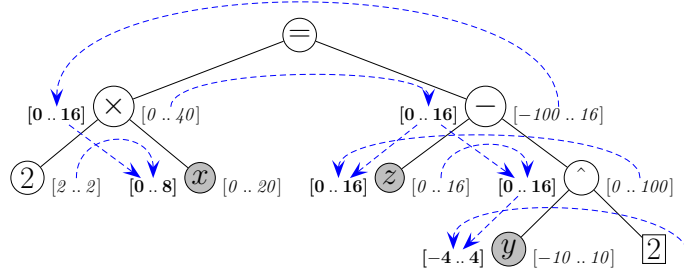


Figure 2: Annotated tree for the backward propagation in the constraint  $2x = z - y^2$

As stated by the following proposition, invoking Algorithm  $\text{HC4}$  on a set of constraints  $\mathcal{C}$  and a box  $\mathbf{B}$  is equivalent to the invocation of Algorithm  $\text{HC3}$  on  $\mathbf{B}$  and the set of constraints generated by the decomposition of the constraints in  $\mathcal{C}$  into primitives:

**Proposition 2.** *Given  $\mathcal{C}$  a set of real constraints and  $\mathbf{B}$  a box, the output box of  $\text{HC4}(\mathcal{C}, \mathbf{B})$  is equal to the one of  $\text{HC3}(\mathcal{C}_{\text{dec}}, \mathbf{B})$ , where  $\mathcal{C}_{\text{dec}} = \bigcup_{c \in \mathcal{C}} c_{\text{dec}}$ .*

*Proof.* The proof is done by noticing that  $\text{HC4}$  and  $\text{HC3}$  use the same narrowing operators, and that the node attributes in  $\text{HC4revise}$  mimic the new variables in  $\text{HC3}$ . The proof then follows since the application order of the same constraint narrowing operators does not influence the fixed-point (confluency property of propagation algorithms [11]).  $\square$

## 4 Box consistency and related algorithms

Box consistency [2] has been introduced to avoid decomposing constraints, thus tackling the dependency problem for variables with many occurrences. Section 4.1 generalizes its definition and surveys the original method used to enforce it. Some connections between Algorithm HC4revise and box consistency are then stated. Finally, Section 4.2 presents a new algorithm to enforce box consistency that avoids relying on the computationally expensive Newton method when a cheaper method (straight evaluation) is available.

### 4.1 Box consistency

The original box consistency definition [2] is based on  $\mathbb{I}_\square$  and on the natural interval extension of constraints. It is generalized below by permitting the use of other interval extensions, and by parameterizing it with two approximate domains. This last modification allows us to take into account some variations over the original definition such as the one given by Collavizza *et al.* [5], where quasi-zeros are approximated by open intervals while the output result is a closed one. The definition applies to (possible) different interval extensions of the original real constraint to define the projections.

**Definition 4.1** (Box $_{\mathcal{A}_1, \mathcal{A}_2, \Gamma}$  consistency). Given  $\mathcal{A}_1$  and  $\mathcal{A}_2$  two approximate domains, let  $c$  be a real  $n$ -ary constraint,  $\Gamma = \{C_1, \dots, C_n\}$  a set of  $n$  interval extensions of  $c$ ,  $k$  an integer in  $\{1, \dots, n\}$ , and  $\mathbf{D} = D_1 \times \dots \times D_n$  a Cartesian product of domains. The constraint  $c$  is said *box consistent w.r.t.  $\mathbf{D}$ , a variable  $x_k$  of  $c$ , and  $C_k$*  if and only if:

$$D_k = \text{apx}_{\mathcal{A}_1}(D_k \cap \{r_k \in \mathbb{R} \mid C_k(D_1, \dots, D_{k-1}, \text{apx}_{\mathcal{A}_2}(\{r_k\}), D_{k+1}, \dots, D_n)\}) \quad (1)$$

Moreover,  $c$  is said box consistent w.r.t.  $\Gamma$  and  $\mathbf{D}$  iff Eq. (1) holds for all  $k$  in  $\{1, \dots, n\}$ . A constraint set  $\mathcal{C}$  is said box consistent w.r.t. a Cartesian product of domains  $\mathbf{D}$  whenever every  $n$ -ary constraint  $c \in \mathcal{C}$  is box consistent w.r.t.  $\mathbf{D}$  and a set  $\Gamma$  of  $n$  interval extensions of  $c$ .

It is then worthwhile noting that box $_{\mathbb{I}_\square, \mathbb{I}_\circ, \{C, \dots, C\}}$  consistency—where  $C$  is the natural interval extension of the considered constraint—corresponds to the definition of box consistency as given by Collavizza *et al.* [5], while box $_{\mathbb{I}_\square, \mathbb{I}_\square, \{C, \dots, C\}}$  consistency is equivalent to the original definition [2].

For the sake of clarity, box $_{\mathbb{I}_\square, \mathbb{I}_\square, \{C, \dots, C\}}$  consistency is henceforth shortened to “box $_\circ$  consistency.”

Box $_\circ$  consistency is enforced by Algorithm BC3revise over a  $n$ -ary constraint  $c$  as follows: CNOs  $N_1, \dots, N_n$  (implementing typically an interval Newton method [14]) are associated to the univariate interval constraints  $C_1^B, \dots, C_n^B$  obtained from the interval extension  $C$  of  $c$  by replacing all the variables but one by their domains. Each  $N_k$  reduces the domain of one

variable by computing the leftmost and rightmost *canonical intervals* such that  $C_k^{\mathbf{B}}$  holds (*leftmost and rightmost quasi-zeros*).

As said in [2],  $\text{box}_o$  consistency computation is far more effective than hull consistency computation when dealing with complex constraints involving the same variables many times, since the global processing of these constraints avoids losing some useful information. Nevertheless, finding the leftmost and rightmost quasi-zeros is computationally expensive. Moreover, it must be done for every variable occurring in the constraint. Therefore,  $\text{box}_o$  consistency is generally not the solution of choice when a complex constraint involves many different variables.

Given a variable  $x_k$  occurring once in a constraint  $c$ , let  $c'$  be an equivalent constraint where  $x_k$  is expressed in term of the other variables of  $c$ , and  $C_{x_k}$  the natural interval extension of  $c'$ . A property of Algorithm HC4revise can now be stated:

**Proposition 3.** *Let  $c$  be an  $n$ -ary constraint,  $C$  its natural interval extension,  $x_k$  a variable occurring exactly once in  $c$ ,  $\Gamma = \{C, \dots, C_{x_k}, \dots, C\}$  a set of  $n$  interval extensions,  $\mathbf{B}$  a  $n$ -ary box, and  $\mathbf{B}'$  the output box of  $\text{HC4revise}^*(c, \mathbf{B}, \mathbb{U}_{\square})$ . Then,  $c$  is  $\text{box}_{\mathbb{I}_{\square}, \mathbb{I}_{\square}, \Gamma}$  consistent w.r.t.  $\mathbf{B}'$ ,  $x_k$  and  $C_{x_k}$ .*

*Proof.* One may prove by induction on the tree-structured representation of  $c$  that, given  $\mathbf{B}'$  the result of  $\text{HC4revise}^*(c, \mathbf{B}, \mathbb{U}_{\square})$ , the interval  $I'_k$  is equal to  $\text{Hull}_{\square}(\rho_c^{(k)}(\mathbf{B}) \cap I_k)$ . The results then follows from Definition 4.1.  $\square$

The exact characterization of the result provided by the execution of HC4revise is still an open problem but we have the following completeness and accuracy result w.r.t. variables with single occurrences:

**Proposition 4.** *Given an  $n$ -ary constraint  $c$ , a box  $\mathbf{B}$ , and  $x_k$  a variable occurring once, the following inclusions hold:*

$$\text{Hull}_{\square}(\rho_c \cap \mathbf{B}) \subseteq \text{HC4revise}(c, \mathbf{B}, \mathbb{U}_{\square}) \subseteq \text{HC4revise}^*(c, \mathbf{B}, \mathbb{U}_{\square})$$

*Proof.* Inclusion  $\text{Hull}_{\square}(\rho_c \cap \mathbf{B}) \subseteq \text{HC4revise}(c, \mathbf{B}, \mathbb{U}_{\square})$  comes from HC4revise being a CNO (Prop. 1), while the other inclusion is due to HC4revise being HC4revise\* where the attribute nodes are intersected with some domains.  $\square$

## 4.2 BC4: a new algorithm for box consistency

Let BC3 be the algorithm similar to HC3 and HC4, where the CNOs used by the *revise function* implement BC3revise.

On the one hand, Algorithm HC3 is efficient over real constraints whose variables have only one occurrence since then, the decomposition process does not amplify the dependency problem and the interval tightening methods are cheap. On the other hand, Algorithm BC3 is in general more efficient

than HC3 since it is able to cancel the dependency problem over one variable with many occurrences in one constraint while the decomposition in HC3 increases it.

The previous remarks have led us to define BC4, a new propagation algorithm presented by Alg. 5. Let BC4\* be Algorithm BC4 where HC4revise is replaced by HC4revise\*.

Algorithm 5: Algorithm BC4

```

BC4(in  $\mathcal{C}$ : set of constraints; inout  $\mathbf{B}$ : box)
begin
  repeat
     $\mathbf{B}' \leftarrow \mathbf{B}$ 
    do
       $NotFinished \leftarrow \mathbf{false}$ 
      foreach  $c \in \mathcal{C}$  do
         $\mathcal{V}_c^1 \leftarrow \{x \in \text{Var}(c) \mid \text{Multiplicity}(x, c) = 1\}$ 
         $\mathbf{B}'' \leftarrow \mathbf{B}$ 
        HC4revise( $c, \mathbf{B}, \mathbb{U}_\square$ )
         $NotFinished \leftarrow (\mathbf{B} \neq \emptyset \wedge ((\exists I_k \neq I_k'' \wedge x_k \in \mathcal{V}_c^1) \vee NotFinished))$ 
      endforeach
      while ( $NotFinished$ )
        if ( $\mathbf{B} \neq \emptyset$ ) then
          BC3( $\mathcal{C}_p^+, \mathbf{B}$ )
        endif
      until ( $\mathbf{B}' = \mathbf{B}$  or  $\mathbf{B} = \emptyset$ )
    end

```

The set  $\mathcal{C}_p^+$  is the set of projection constraints (univariate constraints obtained by replacing all occurrences of every variable but one in any constraint by the corresponding domains) generated from  $\mathcal{C}$ , where the variable occurs more than once.

**Proposition 5.** *Let  $\mathcal{C}$  be a constraint set,  $\mathcal{C}_p^1 = \{C_x \mid \exists c \in \mathcal{C}: x \in \text{Var}(c) \wedge \text{Multiplicity}(x, c) = 1\}$  the set of univariate interval constraints whose variable has only one occurrence,  $\mathcal{C}_p^+$ , the set of univariate interval constraints whose variable has more than one occurrence,  $\mathbf{B}$  a box, and  $\mathbf{B}'$  the output box of BC4\*( $\mathcal{C}, \mathbf{B}$ ). Then, BC4\* enjoys the same properties than HC4, namely: it is a confluent, complete and terminating algorithm. Moreover,  $\mathcal{C}$  is  $\text{box}_{\mathbb{I}_\square, \mathbb{I}_\square, \mathcal{C}_p^1 \cup \mathcal{C}_p^+}$  consistent w.r.t.  $\mathbf{B}'$ .*

*Proof.* Confluency, completeness and termination are proved in the same way as for Algorithm HC4. As for box consistency, it suffices to note that HC4revise\* enforces  $\text{box}_{\mathbb{I}_\square, \mathbb{I}_\square, \mathcal{C}_p^1}$  consistency for variables occurring only once, while BC3 enforces  $\text{box}_{\mathbb{I}_\square, \mathbb{I}_\square, \mathcal{C}_p^+}$  consistency for variables occurring more than once. Consequently, computing the common fixed point of both leads to the result.  $\square$

As a consequence of Prop. 4, the result of applying Algorithm BC4 may be proved complete and included in the result of BC4\*.

## 5 Experimental results

The propagation algorithms HC3, BC3 and BC4 have been implemented and tested on various examples from numerical analysis and CLP(Intervals) benchmarks [14, 6].

Table 5 presents the computational results obtained on a Sun Ultra-Sparc 2/166MHz for algorithms BC3 and BC4 and on an AMD K6/166MHz for HC3, then scaled to the first machine. Propagation algorithms are embedded in a more general bisection algorithm which splits the domains if the desired precision (width of domains) of  $10^{-8}$  is not reached. Neither *improvement factor* nor weakening of box consistency (box<sub>φ</sub> consistency [6]) were used. Computation times exceeding one hour are replaced by a question mark.

Table 1: Experimental results

<i>Benchmark</i>	BC4 ( <i>s.</i> )	BC3 ( <i>s.</i> )	BC3/BC4	HC3 ( <i>s.</i> ) <sup>†</sup>	HC3/BC4
Cosnard 10	0.15	2.45	<b>15</b>	4.10	<b>27</b>
Cosnard 20	0.95	16.50	<b>17</b>	172.75	<b>182</b>
Cosnard 40	2.90	123.30	<b>42</b>	?	↗
Cosnard 80	13.70	916.95	<b>67</b>	?	↗
Broyden 10	2.40	2.35	<b>1</b>	?	↗
Broyden 160	86.65	86.55	<b>1</b>	?	↗
Kearfott 10	0.50	1.40	<b>3</b>	0.75	<b>1.5</b>
Kearfott 11	0.70	6.45	<b>9</b>	0.70	<b>1</b>
i4	32.65	160.80	<b>5</b>	?	↗
bifurcation	2.85	31.10	<b>11</b>	3.55	<b>1.2</b>
DC circuit	0.40	12.15	<b>30</b>	0.65	<b>1.6</b>
pentagon	1.10	12.95	<b>12</b>	0.70	<b>0.6</b>
pentagon all	44.60	2340.00	<b>52</b>	124.20	<b>2.8</b>

<sup>†</sup> Time on an AMD K6/166 scaled to a SUN Solaris 2/166.

“Cosnard  $x$ ” is the Moré-Cosnard problem obtained by discretizing a nonlinear integral equation which is composed of  $x$  equations over  $x$  variables, every variable appearing in every constraint; “Broyden  $x$ ” is the Broyden-Banded problem with  $x$  variables which all have multiple occurrences in the constraints; “Kearfott 10”, “Kearfott 11” and “bifurcation” are respectively some problems from chemistry, kinematics and numerical bifurcation, where the numbers of multiple and simple occurrences of variables in the constraints are approximatively equal; “i4” is a standard benchmark from the interval community; “DC circuit” models an electrical circuit with linear constraints; “pentagon all” describes the coordinates of a set of regular pentagons; “pentagon” is a restriction of “pentagon all” to one pentagon;

the variables in benchmarks “i4”, “DC circuit”, “pentagon” and “pentagon all” have simple occurrences.

**Results analysis:** The well-known behaviours of HC3 and BC3 are confirmed. HC3 is not able to compute all the solutions of the Broyden-Banded problem since the decomposition of constraints amplifies the dependency problem. HC3 is slow for the Moré-Cosnard problem since the decomposition of complex constraints generates numerous primitive constraints and intermediate variables. Consequently, the domains propagation takes a long time. On the same problems, BC3 is far more efficient. HC3 outperforms BC3 on the other problems whose constraints have single occurrences of variables.

The use of HC4revise before BC3 in BC4 greatly accelerates the computations (see Column BC3/BC4). The worst case corresponds to the Broyden-Banded problem composed of multiple occurrences of variables. BC3 does not contract any domain but the overhead is insignificant due to the low cost of HC4revise. The good results for the last benchmarks are not surprising and come from the superiority of BC4 (and HC3) over BC3. The Moré-Cosnard problem is efficiently handled since HC4 is able to contract the domains of variables having multiple occurrences and then HC4 is always used instead of the interval Newton method (in BC3) which computes a sequence of intervals at a greater cost.

Last, BC4 outperforms HC3 on most of the benchmarks (see column HC3/BC4). On the one hand this comes from the superiority of BC3 over HC3 (for example on the Moré-Cosnard and Broyden-Banded problems). On the other hand, HC4 accelerates HC3 for the last benchmarks though the difference is small. A pathological case is “pentagon” which illustrates that a different propagation strategy may lead to different computation times. However the result “pentagon all” is in favour of BC4 though the benchmark is similar.

## 6 Conclusion

The contribution of this paper is twofold: first, an algorithm to enforce hull consistency without decomposing constraints into primitives has been presented; second, an extended definition for box consistency has been given: the original definition [2] only relies on a closed interval approximate domain and on natural interval extension, while the new one is parameterized by two approximate domains and permits defining box consistency for each variable of a constraint according to a different interval extension. As a result, the new definition captures several slightly different definitions of box consistency present in the literature and allows us to devise a new algorithm to efficiently enforce box consistency that may replace both the traditional algorithm used to enforce the original box consistency and the traditional algorithm to enforce hull consistency.

A variation of hull-consistency, *3B consistency*, has been proved to be

more precise than box consistency [5], though computationally more expensive to enforce since it relies on decomposition of constraints into primitives. A promising direction for future research is to reuse Algorithm HC4 to devise a new scheme for enforcing 3B consistency that might compete with BC4 in both speed and accuracy.

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